Properties of Weak Forms of Open Sets in Topological Spaces

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Abstract

This paper attempts to study **open sets described with in topological spaces**and their properties on the opensets with**weak forms.**In topological spaces, Njastad, , 1965 introduced the notion of α-sets in topological space. In 1983, Mashhour et al. introduced, with the help of α-sets, a weak form of continuity which they termed as α-continuity. Since then it has been widely investigated in the literature (see. In 1980, Maheshwari and Thakur, introduced the irresoluteness of α-functions in topological spaces. Recently, continuity and irresoluteness of functions in topological spaces have been researched by many mathematicians and quantum physicists (see , , , ,). For a subset A of X, the closure of A and the interior of A will be denoted by Cl(A) and I nt(A), respectively. Let (X,τ) be a space and A a subset of X.

A point x ∈ X is called a condensation point of A if for each U ∈ τ with x ∈ U, the set U ∩ A is uncountable. A is said to be ω-closed if it contains all its condensation points. The complement of an ω-closed set is said to be ω-open. It is well known that a subset W of a space (X,τ) is ω-open if and only if for each x ∈ W, there exists U ∈ τ such that x ∈ U and U − W is countable. The family of all ω-open sets of a space (X,τ), denoted by τω or ωO(X), forms a topology on X finer than τ. The ω-closure and ω-interior, that can be definedin the same way as Cl(A) and I nt(A), respectively, will be denoted by Clω(A) and I ntω(A), respectively. Several characterizations of ω-closed sets were provided in .Every open set is α-open but the converse need not be true; the intersection of two α-open sets is α-open and the arbitrary union of α-open sets is α-open set, that is, the collection of all α-open sets in X forms a topology on X. The set M is α-open in X if and only if there exists α-open set A in X such that $A \subset M \subset int(cl(A))$; if A is α-open set in X and $A \subset Y \subset X$, then A is α-open in Y; if $A \subset Y$ and Y is α-open subset of X, then A is α-open in Y if and only if A is α-open in X; for more details see , , , .Recall that a map $f{:}X \rightarrow Y$ is called *α-continuous* if $f^{-1}(U)$ is α-open set in X for any open sets U in Y. Every continuous map is α-continuous but the converse not necessarily true; A map $f{:}X \rightarrow Y$ is α-

continuous if and only if $f^{-1}(F)$ is α-closed set in X for any closed sets F in Y; the cartesian product of two α-continuous maps is α-continuous. A map $f:X \rightarrow Y$ is called *α-open* (resp. *α-closed*) map if the direct image of open set (resp. closed) in X is α-open (resp. α-closed) set in Y.

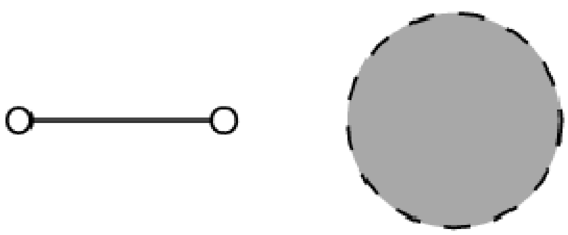
Key words: Topological manifolds, groupoid, Open sets, α-Continuous, α-Irresolute, weak forms.

Introduction

In the set of all real numbers, one has the natural Euclidean metric; that is, a function which measures the distance between two real numbers: $d(x, y) = |x - y|$. Therefore, given a real number x, one can speak of the set of all points close to that real number; that is, within ε of x. In essence, points within ε of x approximate x to an accuracy of degree ε. Note that $\varepsilon > 0$ always but as ε becomes smaller and smaller, one obtains points that approximate x to a higher and higher degree of accuracy. For example, if $x = 0$ and $\varepsilon = 1$, the points within ε of x are precisely the points of the interval $(-1, 1)$; that is, the set of all real numbers between −1 and 1. However, with $\varepsilon = 0.5$, the points within ε of x are precisely the points of $(-0.5, 0.5)$. Clearly, these points approximate x to a greater degree of accuracy than when $\varepsilon = 1$.

The previous discussion shows, for the case $x = 0$, that one may approximate x to higher and higher degrees of accuracy by defining ε to be smaller and smaller. In particular, sets of the form $(-\varepsilon, \varepsilon)$ give us a lot of information about points close to $x = 0$. Thus, rather than speaking of a concrete Euclidean metric, one may use sets to describe points close to x. This innovative idea has far-reaching consequences; in particular, by defining different collections of sets containing 0 (distinct from the sets $(-\varepsilon, \varepsilon)$), one may find different results regarding the distance between 0 and other real numbers. For example, if we were to define R as the only such set for "measuring distance", all points are close to 0 since there is only one possible degree of accuracy one may achieve in approximating 0: being a member of R. Thus, we find that in some sense, every real number is distance 0 away from 0. It may help in this case to think of the measure as being a binary condition: all things in R are equally close to 0, while any item that is not in R is not close to 0.

In general, one refers to the family of sets containing 0, used to approximate 0, as a neighborhood basis; a member of this neighborhood basis is referred to as an open set. In fact, one may generalize these notions to an arbitrary set (X); rather than just the real numbers. In this case, given a point (x) of that set, one may define a collection of sets "around" (that is, containing) x, used to approximate x. Of course, this collection would have to satisfy certain properties (known as axioms) for otherwise we may not have a well-defined method to measure distance. For example, every point in X should approximate x to some degree of accuracy. Thus X should be in this family. Once we begin to define "smaller" sets containing x, we tend to approximate x to a greater degree of accuracy. Bearing this in mind, one may define the remaining axioms that the family of sets about x is required to satisfy.



open interval *open disk*

Let S be a subset of a metric space. Then the set S is open if every point in S has a neighborhood lying in the set. An open set of radius r and center $\mathbf{x}_0$ is the set of all points $\mathbf{x}$ such that $|\mathbf{x} - \mathbf{x}_0| < r$, and is denoted $D_r(\mathbf{x}_0)$. In one-space, the open set is an open interval. In two-space, the open set is a disk. In three-space, the open set is a ball.

More generally, given a topology (consisting of a set X and a collection of subsets T), a set is said to be open if it is in T. Therefore, while it is not possible for a set to be both finite and open in the topology of the real line (a single point is a closed set), it is possible for a more general topological set to be both finite and open.

The complement of an open set is a closed set. It is possible for a set to be neither open nor closed, e.g., the half-closed interval $(0, 1]$.

A *groupoidG* is a small category consisting of two sets G and O_G, called respectively the set of elements (or arrows) and the set of objects (or vertices) of the groupoid, together with two maps $\mu, \beta{:}G \to O_G$, called respectively the *source* and*target* maps of groupoid, the map $\varepsilon{:}O_G \to G$ which is defined by $\varepsilon(x) = 1_x$, where 1_x is called the identity element at x in O_G and ε is called the *object* map, and the *partial multiplication* map $\gamma{:}(G \times G)_{\mu=\beta} \to G$which is defined by $\gamma(g, h) = gh$, where $(G \times G)_{\mu=\beta} = \{(g, h) \in G \times G{:}\mu(g) = \beta(h)\}$. Theses terms must satisfy the following axioms:

G1.$\mu(gh) = \mu(h)$ and $\beta(gh) = \beta(g)$,

G2. $(gh)k = g(hk)$,

G3. $\mu(1_x) = \beta(1_x) = x$ for all $x \in O_G$,

G4. $g1_{\mu(g)} = g$ and $1_{\beta(g)}g = g$,

G5. $g^{-1}g = 1_{\mu(g)}$ and $gg^{-1} = 1_{\beta(g)}$

for all $g, h, k \in G$.

Objective:

This paper intends to exploreand analyze the open sets with weak formsthat provides a means to distinguish two points. If about one of two points in a topological space, there exists an open set not containing the other (distinct) point, the two points are referred to as topologically distinguishable on their weak forms.

Weak forms of open sets : definition

For a groupoid G:

1.

σ: $G \to G$ is the *inversion* map of G defined by $\sigma(g) = g^{-1}$ which is a bijective,

2.

δ: $(G \times G)_{\mu} \to G$ is the *difference* map of G defined by $\delta(h, g) = gh^{-1}$, where$(G\times G)\mu=\{(g,h)\in G\times G{:}\mu(h)=\mu(g)\}$, 3.

π: $G \to O_G \times O_G$ is a map, defined by $\pi(g) = (\mu(g), \beta(g))$.

For a groupoid G and $x, y \in O_G$, we denote the star of G at x by $st_G x$ of the fiber $\mu^{-1}(x) = \{g \in G{:}\mu(g) = x\}$, the co-star of G at y by $cost_G y$ of the fiber $\beta^{-1}(y) = \{g \in G{:}\ \beta(g) = y\}$ and $G(x,y) = st_G x \cap cost_G y$.

Definition

Let G be a groupoid. A *subgroupoid* of G is a pair of subset $N \subset G$ and $O_N \subset O_G$ such that $\mu(N) \subset O_N$, $\beta(N) \subset O_N$, $1_x \in N$ for all $x \in O_N$, and N is closed under partial multiplication and inversion maps in G.

A subgroupoid N of G is *wide* if $O_G = O_N$. A topological groupoid G is a groupoid G together with topologies on G and O_G such that the structure maps of G are continuous, that is; the source map μ, the target map β, the object map ε, the inversion map σ, and the partial multiplication map γ are continuous.

Properties of continuity in topological spaces

Now we investigate some properties of α-continuous maps given to structure maps of a groupoid.

Theorem 3.1

Let G be a groupoid in which G and O_G have topologies. If the inversion map σ is continuous, then the source map μ is α-continuous if and only if the target map β is α-continuous.

Proof

Suppose μ is α-continuous and U be open subset of O_G. Then $\mu^{-1}(U)$ is α-open in G which implies that there exists open subset B of G such that $B \subset \mu^{-1}(U) \subset int(cl(B))$. Since σ is a continuous, we get from its definition that it is a homeomorphism. Henceσ-1(B)⊂σ-1(μ-1(U))⊂σ-1=int.

Then $\sigma^{-1}(\mu^{-1}(U))$ is α-open subset of G. Since G is groupoid, then $\beta = \mu \circ \sigma$. Hence $\beta^{-1}(U) = \sigma^{-1}(\mu^{-1}(U))$. So, $\beta^{-1}(U)$ is α-open subset of G. That is, β is α-continuous.

Conversely, suppose β is α-continuous and V open subset of O_G. Since σ is homeomorphism and $\mu = \beta \circ \sigma$, then $\mu^{-1}(V) = \sigma^{-1}(\beta^{-1}(V))$ is α-open subset of G. Therefore μ is α-continuous. □

Theorem 3.2

Let G be a groupoid in which G and O_G have topologies. If the inversion map σ is continuous, then the partial multiplication map γ is α-continuous if and only if the difference map δ is α-continuous.

Proof

Since the identity map I: Similar for$G \to G$ and the inversion map σ are homeomorphisms, then the map $I \times \sigma: G \times G \to G \times G$, defined by $(I \times \sigma)(g, h) = (g, h^{-1})$, is also homeomorphism. If $(g, h) \in (G \times G)_\mu$ and since G is groupoid, then $\mu(g) = \mu(h) = \beta(h^{-1})$, that is $(g, h^{-1}) \in (G \times G)_{\mu=\beta}$. Hence the restriction map $r:(G \times G)_\mu \to (G \times G)_{\mu=\beta}$ of $I \times \sigma$ on $(G \times G)_\mu$ is homeomorphism. Now suppose the partial multiplication map γ is α-continuous and let U be open subset of G. Then $\gamma^{-1}(U)$ is α-open subset of $(G \times G)_{\mu=\beta}$. This implies that there exists open subset B of $(G \times G)_{\mu=\beta}$ such that $B \subset \gamma^{-1}(U) \subset int(cl(B))$. Hencer-1(B)⊂r-1(γ-1(U))⊂r-1=int.

That is, $r^{-1}(\gamma^{-1}(U))$ is α-open subset of $(G \times G)_\mu$. Since G is a groupoid, then $\delta = \gamma \circ r$. Hence $\delta^{-1}(U) = r^{-1}(\gamma^{-1}(U))$ is α-open subset of $(G \times G)_\mu$, that is, the difference map δ is α-continuous.

Conversely, suppose the difference map δ is α-continuous and let V be open subset of G. If $(g, h) \in (G \times G)_{\mu=\beta}$ and since G is groupoid, then $\mu(g) = \beta(h) = \mu(h^{-1})$, that is $(g, h^{-1}) \in (G \times G)_\mu$. Hence the restriction map $k:(G \times G)_{\mu=\beta} \to (G \times G)_\mu$ of $I \times \sigma$ on $(G \times G)_{\mu=\beta}$ is homeomorphism. Since V is open subset of G, then $\delta^{-1}(V)$ is α-open subset of $(G \times G)_\mu$. Hence $k^{-1}(\delta^{-1}(V))$ is α-open subset of $(G \times G)_{\mu=\beta}$. Since G is groupoid, then $\gamma = \delta \circ k$. Hence $\gamma^{-1}(V) = k^{-1}(\delta^{-1}(V))$ is α-open subset of $(G \times G)_{\mu=\beta}$, that is, the partial multiplication map γ is α-continuous. □

In the following theorem, we show the relation between the partial multiplication map and the map $D:(G \times G)_\beta \to G$ which is defined by $D(g, h) = g^{-1}h$.

α-Topological groupoids

In this section, we start by giving the notions of α-topological groupoid and α-topological subgroupoid. Next, we show the role of the density condition to allow α-topological subgroupoid inherited properties from α-topological groupoid and study the irresoluteness property for the structure maps in α-topological groupoid.

Definition 4.1

A α-topological groupoid G is a groupoid G together with topologies on G and O_G such that the inversion map σ is continuous and the remainder structure maps of G are α-continuous. That is; the source map μ, the target map β, the object map ε, and the partial multiplication map γ are α-continuous.

Note that every topological groupoid is α-topological groupoid but the converse need not be true since any α-continuous need not be continuous

Let G be α-topological groupoid. Then the object map ε is a bijective and α-continuous onto $A = \{1_x : x \in O_G\}$.

Proof

Since G is α-topological groupoid, then the object map $\varepsilon : O_G \to G$ is α-continuous. Define the restriction map $r : O_G \to A$ by $r(x) = 1_x$, for all $x \in O_G$. Then for $x, y \in O_G$,r(x)=r(y)⇒1x=1y⇒μ(1x)=μ(1y)⇒x=y,that is, r is injective. For $1_x \in A$, there exists $x \in O_G$ such that $r(x) = 1_x$, that is, r is surjective. Hence r is bijective. Now suppose U is open set of A. Then $U = V \cap A$, where V is open set of G, but $r^{-1}(U) = \varepsilon^{-1}(V) \cap O_G = \varepsilon^{-1}(V)$. Therefore r is α-continuous. □

From Theorem above, if the object $\varepsilon : O_G \to G$ is α-open map, then the restriction r is α-open map. This follows from the fact that for any open subset U of O_G, then $\varepsilon(U)$ is α-open subset of G but since $\varepsilon(U) \subset A$ and $r(U) = \varepsilon(U) \cap A = \varepsilon(U)$, this implies that $r(U)$ is α-open set in G. Then, there exists an open subset O of G such that $O \subset r(U) \subset int(cl(O))$. This implies that $O \cap A \subset r(U) \cap A \subset int(cl(O)) \cap A$. Since $O \subset r(U) \subset A$, then $O \subset r(U) \subset int_A$, that is, $r(U)$ is α-open subset of A. Therefore, r is α-open map.

Analogous to the pervious explanation, the following theorem deals with a new case of a set of identities in α-topological groupoids.

Fibers of α-topological groupoid

In this section, we give some results about the fibers of α-topological groupoids.

Theorem

Let G be α-topological groupoid. If the singleton $\{g\}$ is open subset of G such that $g \in G(x, y)$, then

1.

The left translation $L_g : cost_G x \to cost_G y$, which is defined by $L_g(h) = gh$, is bijective, α-continuous and α-open.

2.

The right translation $R_g : st_G y \to st_G x$, which is defined by $R_g(t) = tg$, is bijective, α-continuous and α-open.

Proof

Since G is α-topological groupoid, then the multiplication map $\gamma:(G \times G)_{\mu=\beta} \to G$ is α-continuous. So, the restriction $R:\{g\} \times cost_G x \to cost_G y$ of γ, which is defined by $R(g, t) = gt$, is bijective and α-continuous. This follows from: Let $(g, t) \in \{g\} \times cost_G x$. Then $\mu(g) = \beta(t) = x$. And since $\beta(gt) = \beta(g) = y$, then $gt \in cost_G y$. So $R(\{g\} \times cost_G x) \subset cost_G y$. If $t_1, t_2 \in cost_G x$ and $R(g, t_1) = R(g, t_2)$, then $gt_1 = gt_2$, implies, $g^{-1}gt_1 = g^{-1}gt_2$, implies, $1_{\mu(g)}t_1 = 1_{\mu(g)}t_2$. But $\mu(g)\beta(t)$, that is, $t_1 = t_2$. Hence R is injective. Also for $b \in cost_G y$ and $g \in G(x, y)$, $g^{-1}b \in cost_G x$. So, $R(g, g^{-1}b) = (gg^{-1})b = b$. That is, R is bijective. Now let V be open subset of $cost_G y$. That is, there exists open subset U of G such that $V = U \cap cost_G y$. Then $R^{-1}(V) = \gamma^{-1}(U) \cap (\{g\} \times cost_G x)$. Since $\{g\}$ is open in G and $\{g\} \times cost_G x = (\{g\} \times G) \cap (G \times G)_{\mu=\beta}$, then $\{g\} \times cost_G x$ is open subset of $(G \times G)_{\mu=\beta}$. Since γ is α-continuous, then $\gamma^{-1}(U)$ is α-open in $(G \times G)_{\mu=\beta}$, that is, $R^{-1}(V)$ is α-open in $\{g\} \times cost_G x$. Hence R is α-continuous. Therefore, its easy to see that the map $M:cost_G x \to \{g\} \times cost_G x$, which is defined by $M(t) = (g, t)$, is a homeomorphism andLg=R∘M:costGx→M{g}×costGx→RcostGy,is a bijective and α-continuous.

To prove that L_g is α-open, it is enough to prove its inverse Lg-1:costGy→costGx, which is defined by Lg-1(h)=g-1h, is α-continuous. Similarly, we can write Lg-1 as a composite:Lg-1=m∘q:costGy→q(t)=(g-1,t){g-1}×costGy→m(g-1,t)=g-1tcostGx.

Conclusion

open sets are a generalization of open intervals in the real line. In a metric space—that is, when a distance function is defined—open sets are the sets that, with every point *P*, contain all points that are sufficiently near to *P* (that is, all points whose distance to *P* is less than some value depending on *P*). More generally, one defines open sets as the members of a given collection of subsets of a given set, a collection that has the property of containing every union of its members, every finite intersection of its members, the empty set, and the whole set itself. A set in which such a collection is given is called a **topological space**, and the collection is called a topology. These conditions are very loose, and allow enormous flexibility in the choice of open sets. For example, *every* subset can be open (the **discrete topology**), or no set can be open except the space itself and the empty set (the indiscrete topology).In practice, however, open sets are usually chosen to provide a notion of nearness that is similar to that of metric spaces, without having a notion of distance defined. In particular, a topology allows defining properties such as continuity, connectedness, and compactness, which were originally defined by means of a distance.

The most common case of a topology without any distance is given by manifolds, which are topological spaces that, *near* each point, resemble an **open set** of a Euclidean space, but on which no distance is defined in general. Less **intuitive topologies** are used in other branches of mathematics; for example, the Zariski topology, which is fundamental in algebraic geometry and scheme theory.

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